Hierarchical Bayesian spatio-temporal models for mass imputation of CPI in small areas from sparse survey data

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2026 COPAFS Quaterly Meeting

March 13, 2026

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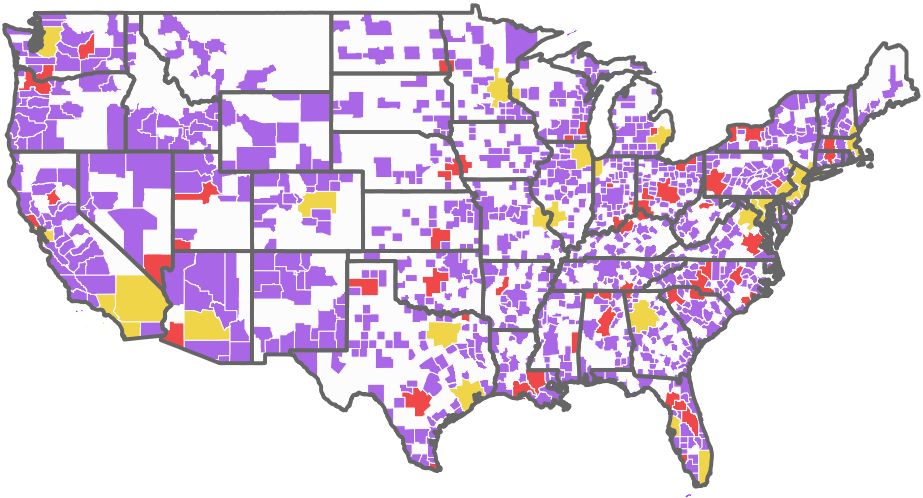
Outline

CPI program overview

- CPI-U population covers 93% of the U.S. population
- BLS collects prices for $\approx 100,000$ goods and services paid by urban households in core-based statistical areas (CBSA) (BLS Handbook of Methods)
- BLS calculates 7,776 basic indexes series $I^c_{a[t]}$ for
  - a - 32 geographic index areas in contiguous USA (9 Census Divisions + 23 large CBSA)
  - c - 243 commodity items
  - t - month of data collection
- Indexes are normalized to 100 starting from a base period (originated in 1983 for most items)
- Percent change (PC) over the last $p = 1$ or $p = 12$ months

$$\left(\frac{I^c_{a[t]}}{I^c_{a[t-p]}} - 1 \right) * 100\%$$

Sparse geographic coverage of the CPI survey



Outline

CBSA are building blocks for SAE

- $\hat{Y}_j, \hat{V}_j$ sampling estimates and sampling errors in $j \in 1, \ldots, n = 73$ sampled CBSA in contiguous USA
- Model estimates $\hat{\theta}_i$ in $i \in 1, \ldots, N = 894$ population CBSA
- $\hat{\theta}_i$ are building blocks for aggregation of any domain d

$$\hat{\theta}_d = \frac{\sum_{i \in d} N_i \hat{\theta}_i}{\sum_{i \in d} N_i}$$

N_i - population in CBSA
d - state, Census Division or any aggregation of CBSA

Borrowing strength: fixed effects, space and time

Model mean in all population CBSA: $\theta_i, i \in 1, \ldots, N = 894$

1. FH: Fay-Herriot area-level model $\left(\Sigma_p \in \mathbb{R}^{P \times P}, \sigma_u^2\right)$:

$$\theta_i = \mathbf{X}_i \boldsymbol{\beta} + u_i$$

$$(\beta_1, \ldots, \beta_P) \sim \mathcal{N}(0, \Sigma_p),\ (u_1, \ldots, u_N) \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_u^2 \mathbf{I}),\ \Sigma_p \in \mathbb{R}^{P \times P}$$

2. CS: Cross-sectional Spatial model $\left(\Sigma_p \in \mathbb{R}^{P \times P}, \Sigma_s \in \mathbb{R}^{N \times N}\right)$:

$$\theta_i = \mathbf{X}_i \boldsymbol{\beta} + u_i,$$

$$(\beta_1, \ldots, \beta_P) \sim \mathcal{N}(0, \Sigma_p),\ (u_1, \ldots, u_N) \sim \mathcal{N}(0, \Sigma_s),\ \Sigma_p \in \mathbb{R}^{P \times P}, \Sigma_s \in \mathbb{R}^{N \times N}$$

3. TS6 and TS12: Spatio-Temporal models, $t \in 1, \ldots, T = 6$ or 12 months:

$$\theta_{it} = \mathbf{X}_i \boldsymbol{\beta_t} + u_{it},$$

$$(u_{11}, \ldots, u_{1T}, u_{21}, \ldots, u_{NT}) \sim \mathcal{N}\left(0, \Sigma_{s \times t}\right),\ \Sigma_{s \times t} = \Sigma_s \otimes \Sigma_t \in \mathbb{R}^{NT \times NT}$$

$$(\beta_{11}, \ldots, \beta_{1T}, \beta_{21}, \ldots, \beta_{PT}) \sim \mathcal{N}\left(0, \Sigma_{p \times t}\right),\ \Sigma_{p \times t} = \Sigma_p \otimes \Sigma_t \in \mathbb{R}^{PT \times PT}$$

Outline

Spatial model: basic formulation

$\mathbf{W} = (W_1, \ldots, W_N)^T$ - field of spatially correlated random effects in CBSA
$i \in 1, \ldots, N$

Spatial correlations depend on distances s_{ij} between centroids of CBSA (i, j)
$\mathbf{A}_{ij} = s_{ij}^{-1}$ - spatial proximity matrix
$\mathbf{Q} = \text{diag}(\mathbf{A1}) - \mathbf{A} \in \mathbb{R}^{N \times N}$ - precision matrix

Model
$\theta_i = \mathbf{X}_i^T \boldsymbol{\beta}_x + \mathbf{W}_i$ - fixed effects + field of random effects
$p(\mathbf{W}|\tau) \propto \tau^{\text{rank}(\mathbf{Q})/2} \exp\left(-\frac{\tau}{2}\mathbf{W}^T\mathbf{Q}\mathbf{W}\right)$ - ICAR prior

Model dimension $D = N + P + 1$

Confounding between fixed and spatial effects

$C(\mathbf{X})$ - space of fixed effects with basis $\mathbf{K} \in \mathbb{R}^{N \times P}$
$C(\mathbf{X})^{\perp}$ - orthogonal space with basis $\mathbf{L} \in \mathbb{R}^{N \times (N-P)}$

$\mathbf{W}_i = \mathbf{K}_i \boldsymbol{\gamma} + \mathbf{L}_i \boldsymbol{\delta}$ - field of random effects decomposed

$\theta_i = \mathbf{X}_i^T \boldsymbol{\beta}_x + \mathbf{L}_i \boldsymbol{\delta}$ - model without confounding between $\mathbf{X}$ and $\mathbf{Q}$

$\mathbf{L} = \mathbf{P}^{\perp} = \mathbf{I} - \mathbf{P}$ - orthogonal basis, where $\mathbf{P} = \mathbf{X}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T$ is projection operator

$p(\boldsymbol{\delta}|\tau) \propto \tau^{(N-P)/2} \exp\left(-\frac{\tau}{2}\boldsymbol{\delta}^T\mathbf{Q}^{\perp}\boldsymbol{\delta}\right)$ - orthogonal ICAR prior
$\mathbf{Q}^{\perp} = \mathbf{L}^T\mathbf{Q}\mathbf{L}$ - orthogonal precision matrix

Model dimension $D = N + 1$

Sparse reparameterization of random effects

The reparameterized vector of random effects

1. have much smaller dimension $Q \ll N$
2. include only positive spatial dependence, i.e. attraction

$M_{\mathbf{X}} = \mathbf{P}^{\perp}\mathbf{A}\mathbf{P}^{\perp}$ is the Moran operator *with respect to* $\mathbf{X}$ and the corresponding Moran's I-statistic is

$$I_{\mathbf{X}}(\mathbf{W}, \mathbf{A}) = \frac{N}{\mathbf{1}^T\mathbf{A}\mathbf{1}} \frac{\mathbf{W}^T\left(\mathbf{P}^{\perp}\mathbf{A}\mathbf{P}^{\perp}\right)\mathbf{W}}{\mathbf{W}^T\mathbf{P}^{\perp}\mathbf{W}}.$$

Matrix $\mathbf{M}_S \in \mathbb{R}^{N\times Q}$ of eigenvectors (harmonics) of the Moran operator $M_{\mathbf{X}}$ corresponding to the first Q largest eigenvalues $\lambda_1 > \cdots > \lambda_Q > 0$.

$$\theta_i = \mathbf{X}_i^T\boldsymbol{\beta}_x + \mathbf{M}_{Si}\boldsymbol{\beta}_S$$

$$p\left(\boldsymbol{\beta}_S|\tau\right) \propto \tau^{Q/2}\exp\left(-\frac{\tau}{2}\boldsymbol{\beta}_S^T\mathbf{Q}_S\boldsymbol{\beta}_S\right)$$

$$\mathbf{Q}_S = \mathbf{M}_S^T\mathbf{Q}\mathbf{M}_S$$

Model dimension $D = P + Q + 1$

Temporal correlations with Gaussian Process (GP) priors

$\boldsymbol{\beta}_t$ - model parameters in collection periods $t \in 1, \ldots, T$.
GP priors allow modeling functional dependence $\boldsymbol{\beta}_t \sim \beta\,(t)$

$\boldsymbol{\beta}_t \sim \mathcal{N}\,(0, K\,(t|\alpha, \lambda))$

$K\,(t|\alpha, \lambda)$ - GP kernel function

α - marginal SD, controls the range of variability of β_t
λ - length-scale, controls the frequency of fluctuations

GP standard kernels

Exponential Quadratic

$$K^{\text{EQ}}(t_i, t_j|\alpha, \lambda) = \alpha^2 \exp\left(-\frac{(t_i - t_j)^2}{2\lambda^2}\right)$$

Matern 3/2

$$K^{\text{M32}}(t_i, t_j|\alpha, \lambda) = \alpha^2 \left(1 + \frac{\sqrt{3}|t_i - t_j|}{\lambda}\right) \exp\left(-\frac{\sqrt{3}|t_i - t_j|}{\lambda}\right)$$

Matern 5/2

$$K^{\text{M52}}(t_i, t_j|\alpha, \lambda) = \alpha^2 \left(1 + \frac{\sqrt{5}|t_i - t_j|}{\lambda} + \frac{5|t_i - t_j|^2}{3\lambda^2}\right) \exp\left(-\frac{\sqrt{5}|t_i - t_j|}{\lambda}\right).$$

Ideas for choosing the right kernel

1. Combine diverse kernels, use informative priors for hyper parameters. More flexible, runs very slow

$$\Sigma_{x|s} = K^{\text{M52}}\left(\alpha_{x|s}, \lambda_{x|s,1}\right) + K^{\text{EQ}}\left(\alpha_{x|s}, \lambda_{x|s,2}\right)$$
$$\lambda_{x|s,1} \sim IG(4,1), \bar{\lambda} = \frac{1}{3}$$
$$\lambda_{x|s.2} \sim IG(4,5), \bar{\lambda} = \frac{5}{3}$$
$$\alpha \sim \mathcal{N}^{+}(0,1)$$

2. Use standard kernels with *constant and reasonable* parameters. Less flexible, runs ≈ 80 times faster

$$\Sigma_{x|s} = K^{\text{EQ}}\left(\alpha_{x|s} = 1, \lambda_{x|s,2} = 1\right)$$

3. Depending on the data, Approach #2 may give acceptable results much faster. Need more experimenting.

Outline

Calibration to direct estimates

$(\hat{Y}_d, \hat{V}_d)$ - direct sampling estimates in Census Divisions $d \in 1, \ldots, 9$.

Calibration equation for CBSA estimates $\theta_i,\ i \in d$:

$$\hat{Y}_d \sim N\left(\theta_d, \hat{V}_d^2\right)$$

$$\theta_d = \frac{\sum_{i \in d} N_i \theta_i}{\sum_{i \in d} N_i}$$

Global-local priors for fixed effects

Fixed effects $\mathbf{X} \in \mathbb{R}^{N \times P}$, $\boldsymbol{\beta}_x^p$, $p \in 1, \ldots, P$ - model coefficients

$$\boldsymbol{\beta}_x^p \sim N(0, \sigma_p^2)$$

$$\boldsymbol{\sigma}_p | \sigma_G \sim C^+(0, \sigma_G), \ \sigma_G | \sigma \sim C^+(0, \sigma)$$

$$C^+(x|0, a) = \frac{1}{\pi} \frac{a}{x^2 + a^2}, x > 0 \text{ - half-Cauchy distribution}$$

$$E(\boldsymbol{\beta}_x^p | \mathbf{Y}) = \{1 - E(\kappa_p | y)\} \left(\mathbf{X}^T \mathbf{X}\right)^{-1} \left(\mathbf{X}^T \mathbf{Y}\right)_p$$

$$\kappa_p = 1 / \left(1 + \sigma_p^2\right) \sim \text{Beta}(0.5, 0.5)$$

κ_p - shrinkage coefficient with $\cup$-shaped "horseshoe" prior

Smoothed sampling variances via co-modeling (aka GVF)

Sampling equation of the Fay Herriot model

$$\hat{Y}_j|\theta_j, \hat{V}_j^2 \overset{\text{ind}}{\sim} N\left(\theta_j, \hat{V}_j^2\right) \text{ - no co-modeling}$$

Sampling equations for co-modeling of means and variances.
ν_j^2 - latent variance.

$$\begin{cases} \hat{Y}_j|\theta_j, \nu_j^2 \overset{\text{ind}}{\sim} N\left(\theta_j, \nu_j^2\right) \\ \hat{V}_j^2|a, b, \nu_j^2 \overset{\text{ind}}{\sim} G\left(\frac{an_j^*}{2}, \frac{an_j^*}{2b\nu_j^2}\right), \nu_j^2 \overset{\text{ind}}{\sim} IG\left(2, 1\right) \end{cases}$$

Outline

Fuel price change over 12-month (%)

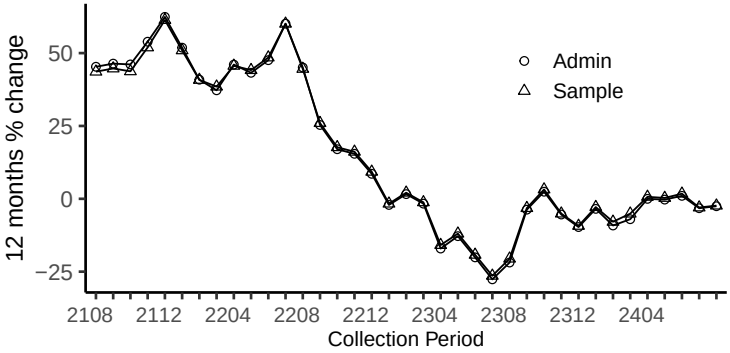


Figure: National average of 12-month change (%) in fuel prices in CBSA estimated from the data of CPI survey (△) and administrative dataset (○) by monthly collection periods from 202107 to 202407.

- Estimation from the CPI survey sample of fuel prices
- Validation against Admin data

Outline

Model fit averaged over collection periods

			Sampled CBSA				Non-sampled CBSA			
Model	Time (s)	WAIC	SD	95% CI	RMSE	R^2	SD	95% CI	RMSE	R^2
FH	60	199	2.6	.91	3.1	.68	3.5	.82	4.7	.08
CS	188	189	2.6	.93	2.4	.75	4.3	.86	4.4	.23
TS6	**921**	**166**	**2.6**	**.97**	**1.9**	**.83**	**5.4**	**.93**	**4.1**	**.31**
TS12	2661	170	2.7	.97	1.9	.83	6.6	.96	4.4	.28

Table: Summary of model fit averaged over collection periods from 202310 to 202407

WAIC - Widely Applicable Information Criteria

$$\bar{y}^{\text{adm}} = \frac{1}{N}\sum_i y_i^{\text{adm}}, \bar{\hat{\theta}}^{\text{mod}} = \frac{1}{N}\sum_i \hat{\theta}_i^{\text{mod}}$$

$$SS_{\text{tot}}^{\text{adm}} = \sum_i \left(y_i^{\text{adm}} - \bar{y}^{\text{adm}}\right)^2,\ SS_{\text{tot}}^{\text{mod}} = \sum_i \left(\hat{\theta}_i^{\text{mod}} - \bar{\hat{\theta}}^{\text{mod}}\right)^2,\ SS_{\text{res}} = \sum_i \left(y_i^{\text{adm}} - \hat{\theta}_i^{\text{mod}}\right)^2$$

$$\text{SD} = \sqrt{SS_{\text{tot}}^{\text{mod}}/N},\ \text{RMSE} = \sqrt{SS_{\text{res}}/N},\ R^2 = 1 - \frac{SS_{\text{res}}}{SS_{\text{tot}}^{\text{adm}}}$$

LOO-CV and WAIC by collection periods

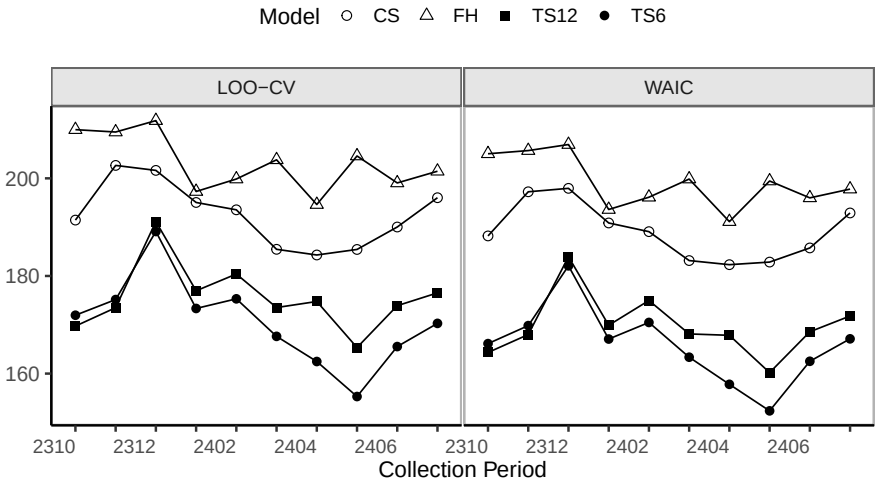


Figure: Leave One Out Cross Validation (LOO-CV) and Widely Applicable Information Criteria for *FH*, *CS*, *TS6* and *TS12* models. Data collection periods from 202310 to 202407.

SD and 95% CI coverage by collection periods

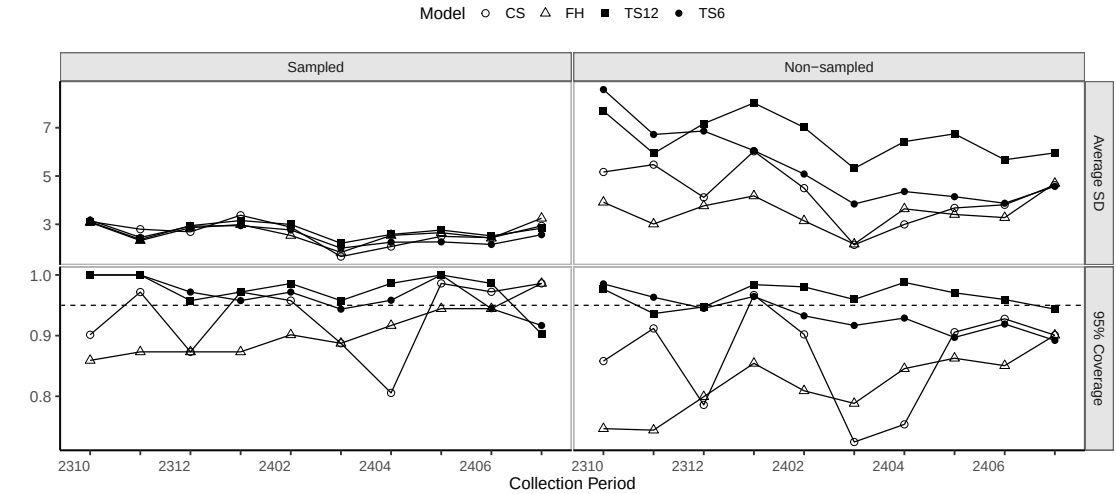


Figure: Average SD and 95% CI coverage rate of the estimates by *FH*, *CS*, *TS6* and *TS12* models for the sampled and non-sampled CBSA. Data collection periods from 202310 to 202407.

RMSE and R^2 by collection periods

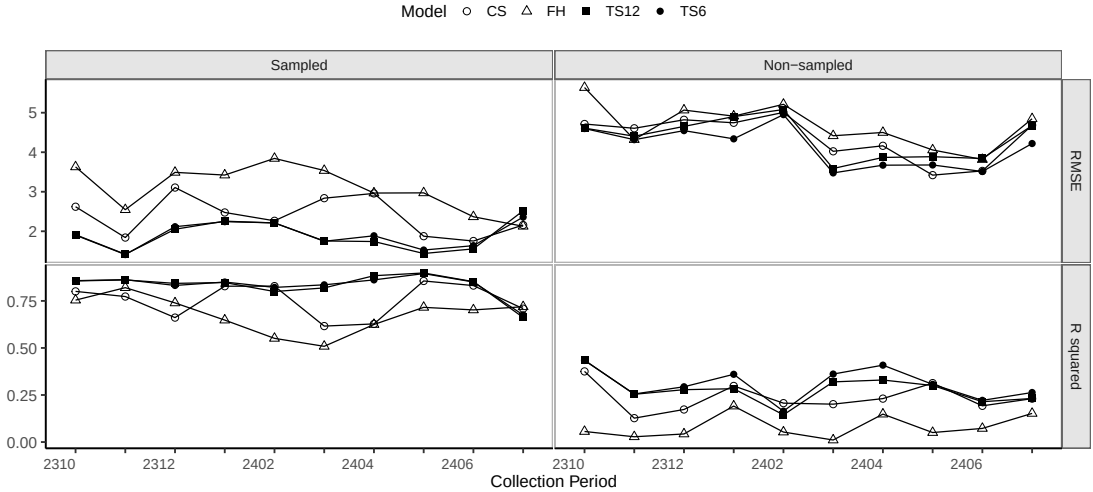


Figure: RMSE and coefficient of determination R^2 of the estimates by models *FH*, *CS*, *TS6* and *TS12* models relative the estimates from the administrative data (○). Data collection from 202310 to 202407.

States: Model vs Admin data

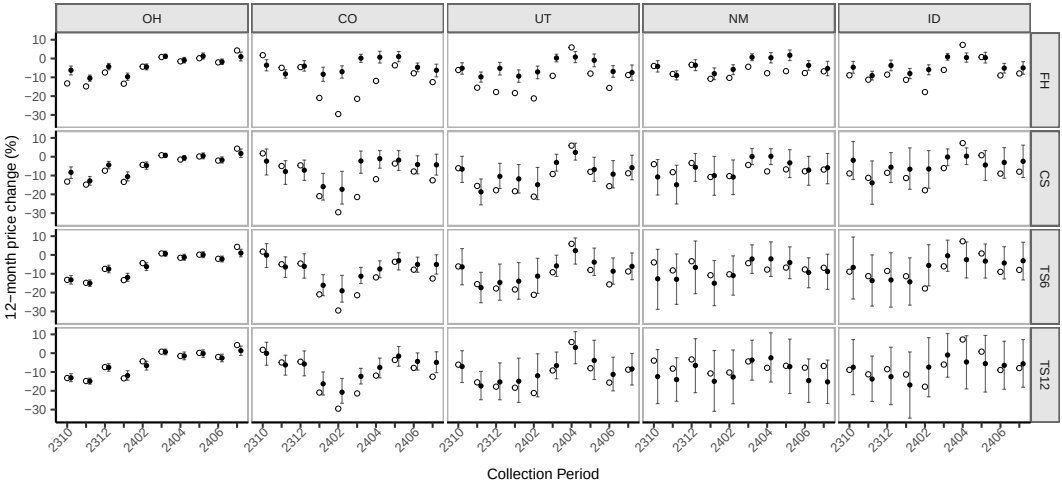


Figure: Model-based estimates (•) in Ohio (OH), Colorado (CO), Utah (UT), New Mexico(NM) and Idaho(ID) with 90% credible intervals by *FH*, *CS*, *TS6* and *TS12* models relative to the "gold-standard" estimates (○).

Census Divisions: Model vs Admin data

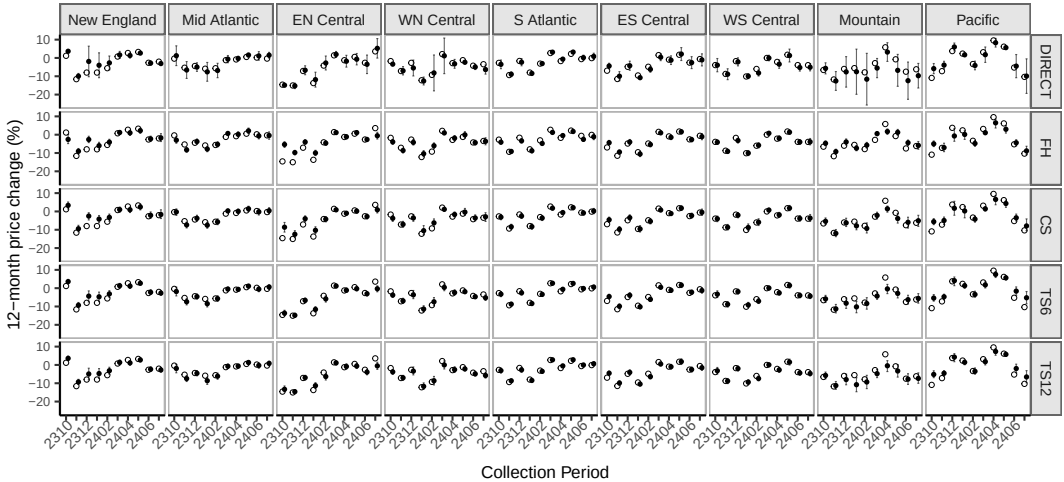


Figure: Model-based estimates (•) in Census Divisions with 90% credible intervals by *FH*, *CS*, *TS6* and *TS12* models relative to the "gold-standard" estimates from administrative data (○).

CBSA: socio-demographic groups

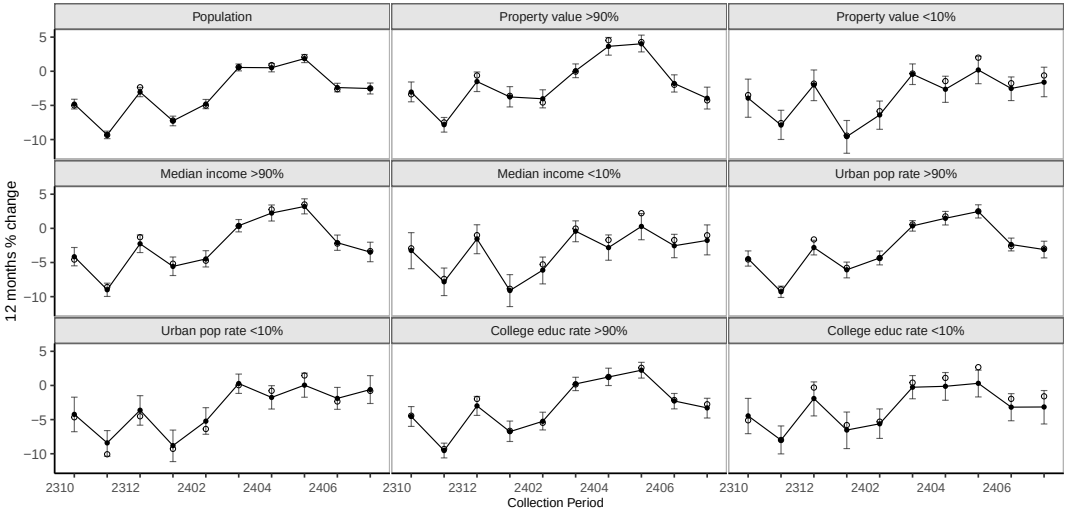


Figure: *TS6* model estimates (●) in socio-demographic groups with 90% credible intervals relative to the "gold-standard" estimates from administrative data (○).

Outline

Conclusions

- Why do spatio-temporal models improve over the Fay-Herriot model?
  - Sparse geography of CBSA sample
  - Low explanatory power of socio-demographic fixed effects
  - Strong spatio-temporal correlations of CPI survey data
- Why *TS6* model is better than *TS12*?
  - Data aggregation over collection periods reduces Bias and increases Variance. Same old Bias vs Variance dilemma
- New analyzes with CBSA building blocks
  - Estimates with proper uncertainty measures in any arbitrary geography: States, Census Divisions, North California, South Florida, North and South Carolinas, etc
  - Define any aggregation of CBSA based on socio-demographic characteristics: low income, high education, high % of manufacturing employment, etc. Make reliable inferences for time series

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Thanks to the colleagues from the Office of Prices and Living Conditions:

Bill Johnson
Jenny Fitzgerald
Steven Paben
Rob Cage
Alex Traczyk
Johanky Reyes